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Topological mass term for gravity induced by matter

J. J. van der Bij,
Robert D. Pisarski,
and Sumathi Rao

*Fermi National Accelerator Laboratory¹
P.O. Box 500, Batavia, Illinois 60510*

Abstract

The contribution of massive fermions and topologically massive gauge fields to a Chern-Simons term for the graviton is computed to one loop order in three dimensions. Invariance under large coordinate transformations places restrictions on the number of these massive matter fields. The parity anomaly for massless fermions is determined, and a possible parity anomaly for massless gauge fields is suggested.



The Chern-Simons mass term for gauge fields in three space-time dimensions has generated much interest.⁽¹⁻⁷⁾ For non-abelian gauge fields, invariance under large gauge transformations requires the mass to be quantized in units of the coupling constant.⁽¹⁻³⁾ To one loop order, massive fermions produce a (finite) renormalization of the Chern-Simons term, which can result in restrictions on the number of fermion fields.⁽³⁻⁶⁾ When the fermions are massless, these restrictions are unchanged, although then they arise from a non-perturbative parity anomaly.⁽⁴⁻⁶⁾

In this letter we consider a Chern-Simons term for gravity in three Euclidean space-time dimensions.^(1,7) Like non-abelian gauge fields, for Euclidean gravity the coupling of the Chern-Simons term must be quantized to preserve invariance under large coordinate transformations.⁽¹⁾ Coupling massive fermions and (topologically) massive spin-1 fields to gravity, the renormalization of the graviton's Chern-Simons term is computed to one loop order. Topological invariance leads to limitations on the number and type of both spin-1/2 and spin-1 fields. For massless fermions, the same result follows from a non-perturbative parity anomaly. In contrast, we are unable to derive a nonperturbative parity anomaly for massless spin-1 fields, but conjecture that such an effect might occur.

The Lagrangian for Einstein gravity is given by

$$L = -\frac{1}{\kappa^2} \sqrt{g} R. \quad (1)$$

As usual, R is the scalar curvature, $g_{\mu\nu}$ the metric tensor, $\sqrt{g} \equiv (\det g_{\mu\nu})^{1/2}$, and κ^2 is Newton's constant. A Chern-Simons term can be added to this Lagrangian,

$$L_{CS} = -\frac{i}{4\kappa^2\mu} \epsilon^{\mu\nu\lambda} (R_{\mu\nu\alpha\beta} \omega_{\lambda}{}^{\alpha\beta} + \frac{2}{3} \omega_{\mu\alpha}{}^b \omega_{\nu b}{}^c \omega_{\lambda c}{}^a). \quad (2)$$

The basic dynamical variables of the theory are the dreibein fields e_{μ}^a , where $g_{\mu\nu} = e_{\mu}^a e_{\nu}^a$ and $e^{a\mu} = g^{\mu\nu} e_{\nu}^a$. From the dreibein, the spin connection $\omega_{\mu}{}^{ab}$ is given by

$$\omega_{\mu}{}^{ab} = \frac{1}{2} e^{a\nu} (\partial_{\mu} e_{\nu}^b - \partial_{\nu} e_{\mu}^b) + \frac{1}{4} e^{a\nu} e^{b\lambda} (\partial_{\lambda} e_{\nu}^c - \partial_{\nu} e_{\lambda}^c) e_{c\mu} - (a \leftrightarrow b). \quad (3)$$

while the curvature tensor is

$$R_{\mu\nu\alpha\beta} = \partial_{\mu} \omega_{\nu\alpha\beta} - \partial_{\nu} \omega_{\mu\alpha\beta} + \omega_{\mu\alpha}{}^c \omega_{\nu cb} - \omega_{\nu\alpha}{}^c \omega_{\mu cb}. \quad (4)$$

For pure Einstein gravity, on the mass shell there are no propagating degrees of freedom. By adding a Chern-Simons term, the graviton becomes a propagating field of spin 2 and mass μ .⁽¹⁾ Under a parity transformation, L_{CS} changes sign, while L does not. As a consequence, for finite μ the graviton has a definite handedness, depending upon the sign of μ .

The action formed from $L + L_{CS}$ is not invariant under arbitrary coordinate transformations. In general, the action transforms into itself, a surface term, plus a term which depends upon the gravitational winding number of the coordinate transformation. We assume that the manifold is compact, so that the surface term vanishes. To evaluate the term involving the winding number, remember that the Chern-Simons term in three dimensions is the fourth component of a topological current, where the divergence of this four-current is proportional to the density of gravitational instantons in four dimensions. Integrating the instanton density over four dimensional space-time gives the instanton number p_{grav} of a gravitational instanton,

$$p_{grav} = \frac{1}{96\pi^2} \int \epsilon^{\mu\nu\lambda\sigma} R_{\mu\nu}{}^{ab} R_{\lambda\sigma ab} \sqrt{g} d^4x; \quad (5)$$

p_{grav} is an integer for all compact orientable four-manifolds. Since the instanton number can be expressed as a difference of (three-dimensional) winding numbers, the normalization of p_{grav} can be used to show that under topologically non-trivial coordinate transformations, the action for topologically massive gravity changes by $(2\pi i)(6\pi/(\mu\kappa^2))$ times an integer. Consequently, invariance under large coordinate transformations imposes a quantization condition that relates the mass μ to Newton's constant:

$$q \equiv \frac{6\pi}{\mu\kappa^2} = \text{an integer}. \quad (6)$$

This quantization condition is analogous to that in non-abelian gauge theories, since the spin connection can be viewed as a gauge field for the group of local Euclidean rotations, which is $SO(3)$.⁽¹⁾ There appears to be no quantization condition on q in Minkowski space-time, for the maximal compact subgroup of the Lorentz group is $SO(2)$, which is homotopically trivial.

We consider the effect of matter fields on topologically massive gravity in weak coupling. Because Newton's constant κ^2 has dimensions of inverse mass, weak coupling means that any mass should be smaller than $1/\kappa^2$. We assume this condition holds for all matter fields as well as for the Chern-Simons mass for the graviton

itself; if $\mu\kappa^2$ is small, the integer q in eq. (6) is a large number.

While the assumption of weak coupling generally means that one can trust the loop expansion, in this case that is not at all obvious, for by power counting the model is not renormalizable. Why then are our calculations of interest? Foremost is the fact that the quantity we compute is ultraviolet finite, at least to leading order. Thus in obtaining constraints on the matter fields, we derive a necessary condition for the self-consistency of topologically massive (quantum) gravity.

Matter fields are coupled to gravity in a standard fashion. The fermion part of the Lagrangian is

$$L_{fer} = e(-\bar{\psi}\gamma^a e_a^\mu D_\mu \psi - m_f \bar{\psi}\psi). \quad (7)$$

ψ is a two component spinor, γ^a are the three dimensional Euclidean gamma matrices, and m_f is the mass of the fermion. The covariant derivative is given by $D_\mu = \partial_\mu + \frac{i}{4}\epsilon_{abc}\gamma^c\omega_\mu^{ab}$. The photon Lagrangian is

$$L_{ph} = -\frac{1}{4}e g^{\mu\nu}g^{\lambda\sigma}F_{\mu\lambda}F_{\nu\sigma} - \frac{i}{2}m_{ph}\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda), \quad (8)$$

where m_{ph} is the (topological) mass of the photon. For the quantity of interest, it doesn't matter if the gauge field is abelian or not, so for simplicity we have taken it to be a photon. Lastly, we assume that there are N_f flavors of fermions and N_{ph} types of photons, identical in all respects.

To leading order it suffices to linearise the dreibein to first order in κ . Writing

$$e_\mu^a = \delta_{a\mu} + \kappa h_{a\mu}, \quad (9)$$

we compute the two point function of the graviton in the presence of fermion and photon loops. The fermion propagator is

$$\Delta(p) = \frac{-i}{\not{p} - im_f}; \quad (10)$$

the fermion-fermion-graviton vertex is

$$\Gamma_{\alpha\beta}(k;p,q) = \kappa[-\delta_{\alpha\beta}(i\not{p} + i\not{k}/2 + m_f) + i\gamma_\beta(p_\alpha + k_\alpha/2) + \frac{1}{4}\epsilon_{\mu\beta\alpha}k_\mu]. \quad (11)$$

α and β are the graviton indices and k its momentum, while p and q are the momenta of the fermions. In Landau gauge the photon propagator is

$$\Delta_{\mu\nu}(p) = \frac{1}{p^2 + m_{ph}^2} \left[\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m_{ph} \epsilon_{\mu\nu\lambda} \frac{p^\lambda}{p^2} \right], \quad (12)$$

with the photon-photon-graviton vertex

$$\begin{aligned} \Gamma_{\alpha\beta;\mu\nu}(k; p, q) = & \kappa [p \cdot q \delta_{\alpha\beta} \delta_{\mu\nu} - \frac{1}{2} \delta_{\alpha\beta} (p_\mu q_\nu + p_\nu q_\mu) - \delta_{\mu\nu} (p_\alpha q_\beta + q_\alpha p_\beta) \\ & + \delta_{\alpha\mu} p_\nu q_\beta + \delta_{\beta\nu} p_\alpha q_\mu + \delta_{\beta\mu} p_\nu q_\alpha + \delta_{\alpha\nu} p_\beta q_\mu - p \cdot q (\delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu})], \end{aligned} \quad (13)$$

where again α, β , and k refer to the graviton, while μ, ν, p , and q are the indices and momenta of the photons.

We wish to compute the renormalized value of q ,

$$q_{ren} = \left(\frac{6\pi}{\mu\kappa^2} \right)_{ren}, \quad (14)$$

to one loop order. For this it is only necessary to compute the parity-odd part of the graviton propagator about zero momentum. Although the total contribution of matter fields to $(\mu\kappa^2)_{ren}$ is ultraviolet finite, individual terms can be divergent. These are regulated by a parity even regulator such as dimensional regularisation. We find

$$q_{ren} = q - \frac{N_f}{16} \text{sign}(m_f) + \frac{N_{ph}}{8} \text{sign}(m_{ph}) + \dots \quad (15)$$

There is an elementary way to view eq. (15). The fermion mass and the Chern-Simons terms for the photon and graviton are all odd under parity. When loop effects are computed, it is natural to expect that the parity odd part of the matter sector will affect that of gravity.

The value of q_{ren} in eq. (15) includes the contribution of all matter fields to one loop order (scalars do not enter), but it does not include the contribution of virtual gravitons. If the example of a non-abelian gauge theory is any guide, this calculation is rather more involved.⁽³⁾ Even so, while virtual gluons do contribute to the quantity analogous to q_{ren} in a non-abelian theory, their contribution is always an integer. Similarly, we assume that virtual gravitons contribute some integer to q_{ren} . This assumption is not as cavalier as it might first appear: it is equivalent to the statement that if a theory of topologically massive gravity (without matter

fields) is invariant under large coordinate transformations classically, it will remain so quantum mechanically.

For the theory to be invariant under large coordinate transformations order by order in the loop expansion, $q_{ren} - q$ must be an integer. For a theory with only fermions or photons all with the same sign of the mass, eq. (15) implies that the number of fermions must be a multiple of sixteen, or the number of photons a multiple of eight. There are obviously many other solutions possible if both fermions and photons with either sign of the mass are allowed.

Are there contributions to q_{ren} at higher order in the loop expansion? On dimensional grounds, to two loop order such terms could only be $O(\mu\kappa^2)$, $O(m_f\kappa^2)$, or $O(m_{ph}\kappa^2)$. In weak coupling, these terms are infinitesimally small, so the only way in which q_{ren} could remain an integer would be if all such corrections vanished identically; whether or not they do is another matter. This would also mean that if ultraviolet infinities appear in μ_{ren} and κ_{ren}^2 , they must cancel when q_{ren} is formed. This would be like the consequences of the topological Ward identity in non-abelian gauge theories.⁽³⁾

The q_{ren} of eq. (15) is computed in weak coupling, when m_f and m_{ph} are much less than $1/\kappa^2$, but the result depends only upon their sign, and is independent of their magnitude. A natural question is then do similar consequences follow when these masses vanish? As explained in ref. (6), by considering the physically relevant order of limits when m_f and m_{ph} are tuned to zero, there will be no contribution to the Chern-Simons term in perturbation theory. As in non-abelian gauge theories, however, there can be a non-perturbative parity anomaly.

We start with the case of massless fermions without photon fields. In this instance, we can copy the treatment of massless fermions in a non-abelian gauge theory.⁽⁴⁾ The essential trick is to introduce a fourth coordinate which interpolates between sectors with different winding number. The three-dimensional fermion determinant is extended to one in four dimensions, and the phase of the (three-dimensional) fermion determinant found from the spectral flow in four dimensions. The spectral flow is summarized by the relation between the number of left- and right-handed zero modes, n_L and n_R , and the instanton number for the gauge field,

p_{gauge} . For fermions in the fundamental representation of an $SU(2)$ gauge field,

$$n_L - n_R = p_{gauge}. \quad (16)$$

From eq. (16), configurations with odd instanton number p_{gauge} have an odd number of zero modes. This means that for a single fermion flavor in three dimensions, the fermion determinant in the presence of a pure gauge field with odd winding number differs from that in zero field by a minus sign. To ensure topological gauge invariance order by order in the loop expansion, the total number of fermion flavors must be even.^(4,5) The restriction is identical to that found for massive fermions, where the result analogous to eq. (15) is $q_{ren} = q + (N_f/2)sign(m_f)$.⁽⁵⁾

This reasoning can be extended to topologically massive gravity, at least naively. All that is needed is the relation for zero modes in the field of a gravitational instanton:⁽⁸⁾

$$n_L - n_R = \frac{1}{8} p_{grav}. \quad (17)$$

From eq. (17) we assume that "one-eighth" of a zero mode exists per unit of instanton number. If so, then per flavor the fermion determinant in three dimensions acquires a phase of $\pi i/8$ for a coordinate transformation of unit winding number. To ensure invariance under arbitrary coordinate transformations, the number of fermion flavors must be a multiple of sixteen — exactly as from eq. (15).

This argument has no rigor whatsoever. After all, there can never be "one-eighth" of a zero mode. The resolution of this problem is well-known. It is only possible to define a spin structure globally on a manifold if the second Stiefel-Whitney class of that manifold vanishes.⁽⁹⁾ By Rohlin's theorem,⁽¹⁰⁾ for manifolds in four dimensions this implies that the instanton number must be a multiple of sixteen. (Note that the Pontryagin number is three times the instanton number p_{grav} of eq. (5).) From eq. (17), manifolds that admit a global spin structure have not only integral but an even number of zero modes.

We nevertheless believe that while our arguments are imprecise, our conclusion is correct — that whatever the mass of the fermion, N_f must be a multiple of sixteen. The crucial point is that a manifold fails to admit a globally defined spin structure because of an essentially abelian phase factor. (This is illustrated by the way Rohlin's theorem is avoided by an extended spin structure,⁽¹¹⁾ where an abelian

Dirac monopole is added to the field content.) Being abelian, it should be sensible to add together the phases of the fermion determinant from different flavors.

This leaves the question of massless gauge fields, but here we are at a loss as to how to proceed. An essential part of the treatment of fermions is that in four dimensions the massless fermion determinant is chiral, so it is possible to define left- and right-handed zero modes. This is not true for gauge fields; indeed, the very concept of a topological mass is manifestly special to three dimensions.

On the other hand, the result for fermions is intuitively "obvious" — since the restriction on the number of matter fields derived in weak coupling is independent of their mass, logically the simplest possibility is that this restriction is unchanged as the mass changes. Otherwise, it would have to change discontinuously as the mass decreased. For this reason, we propose that there is a nonperturbative parity anomaly for massless gauge fields in the presence of a background gravitational field, and that this gives the same restrictions as found from eq. (15). We do not know how in detail this might come about.

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